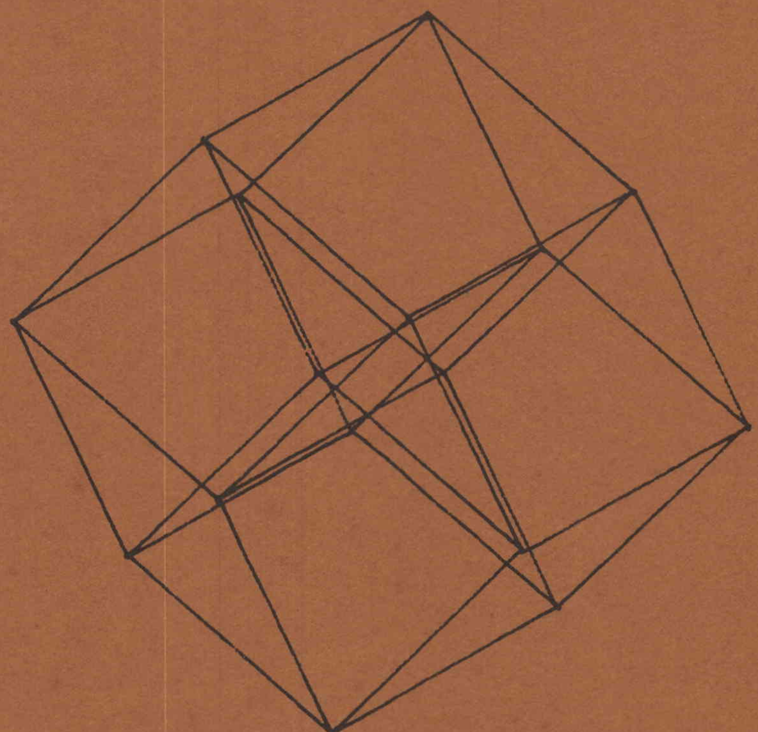
N-DIMENSIONAL ROTATIONS



by Mark P. Gottlieb

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

N - DIMENSIONAL ROTATIONS

One of the main features of computer graphics is the rotation of figures in space. This paper goes into the theory of rotations. It then shows how to apply these theories to rotating any figure stored in the computer in any dimension desired.

We can visualize the first few dimensions--0,1,2 and the 3rd dimension. To clarify this, consider a cube. A cube has width, height, and depth and thus may only exist in the 3rd dimension (or any higher dimension). If a cube were to be in a 2-dimensional world, it would have width and height but no depth. A square is a 2-dimensional cube. If the cube were to be in a 1-dimensional world, it would only be able to have one dimension--commonly called length. A line is a one-dimensional cube. If our cube were to be in a 0-dimensional world, it would have no dimensions at all--a point. A point has no dimensions.

Most persons cannot visualize objects in any dimension above the 3rd dimension. For programming purposes this poses no problem since mathematically we can manipulate any figure in any dimension we like with the aid of a computer. We can also display a figure on a two-dimensional screen or paper. But by doing this we lose much information. By having the object move by making it into a movie film, or using stereo pictures with special lenses or filters, we can get much more information. Better yet, make a stereo movie film and we can visualize 4-dimensional objects almost as though we were in the 4th dimension.

From now on, we will consider rotations in the 2nd, 3rd, and 4th dimensions. One can easily extend the theory learned here to any dimension.

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

When rotating in 2 dimensions, we think of rotating our figure in the X-Y plane around the origin. This is correct. When we move into the 3rd dimension we think of rotation around the X axis or the Y axis or the Z axis. This is also correct but leads to problems when going to dimensions greater than the 3rd dimension. It is better to continue rotating "in a plane." So, in 3 dimensions, rotating around the Y axis is actually rotating in the X-Z plane around the origin and so on.

When we move to the 4th dimension we need to label a new axis--let's call it the Q axis. In the 3rd dimension, each axis is at right angles to the other. The same is true for the 4th dimension as well as greater dimensions. So the X axis, the Y axis, the Z axis, and the Q axis are all perpendicular (at right angles) to each other.

In 2 dimensions we have only 1 plane within which to rotate--the X-Y plane. In 3 dimensions we have 3 planes ($n(n-1)/2 = 3(3-1)/2 = 3$); the X-Y plane, the X-Z plane, and the Y-Z plane within which we can rotate. In the 4th dimension we have 6 planes ($4(4-1)/2 = 6$), the X-Y plane, the X-Z plane, the Y-Z plane, the X-Q plane, the Y-Q plane, and the Z-Q plane within which we can rotate. The 5th dimension would have 10 planes ($5(5-1)/2 = 10$) and so on.

Since one rotates around a point--usually the origin--it is easiest to use polar coordinates to rotate the figure. Once the figure has been rotated to the desired location (rotating a certain number of degrees), it can then be converted back to the X-Y, X-Y-Z, X-Y-Z-Q, or what ever coordinate system was used originally. (Before proceeding further in this paper, be sure to read through the paper in this series on matrices if not at ease with the use of matrices.)

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

First, imagine the entire plane rotating in a counterclockwise direction about the origin while the axes remain fixed. If the rotation is made through an angle θ, and if each point in the plane is expressed in terms of polar coordinates $(r,\emptyset)$, then as Figure 1 shows, each point $(r,\emptyset)$ rotates into the point $(r,\emptyset+\theta)$. If (x,y) represents the Cartesian coordinates of the point with polar coordinates $(r,\emptyset)$, then

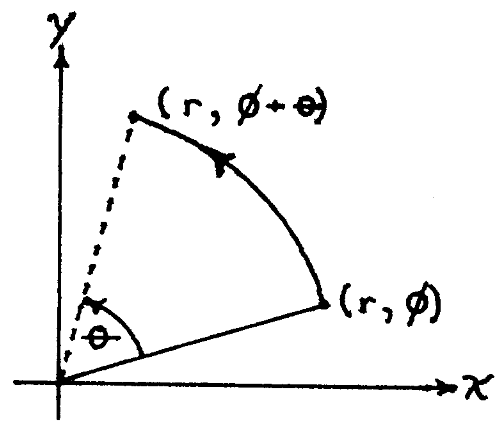

Figure 1

$$(x,y) = (r\cos\emptyset, r\sin\emptyset),$$

and the image (x',y') of (x,y) after the rotation is

$$\begin{aligned}(x',y') &= (r\cos(\emptyset+\theta), r\sin(\emptyset+\theta)) \\ &= (r\cos\emptyset\cos\theta - r\sin\emptyset\sin\theta,\ r\sin\emptyset\cos\theta + r\cos\emptyset\sin\theta).\end{aligned}$$

Then, since $r\cos\emptyset = x$ and $r\sin\emptyset = y$, we have

$$(x',y') = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$$

That is, such a rotation of the axes maps the point (x,y) into the point $(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$, or in terms of vectors, maps the vector $[x,y]$ into the vector $[x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta]$. By inspection,

$$[x',y'] = [x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta] = [x,y]\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}.$$

In other words, the rotational matrix which when multiplied by any point $[x,y]$ will give us our new point, $[x',y']$ is

$$\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}.$$

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

To summarize, the above matrix will rotate a point (x,y) to a new location (x',y') (determined by the degree of revolution--θ) as shown below.

$$\begin{bmatrix} x',y' \end{bmatrix} = \begin{bmatrix} x,y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

To think in general terms, we have rotated a point in a plane around the origin.

Now for n-dimensional rotations!

We now have the necessary knowledge to rotate any dimensional figure we might like. The key to remember is that one rotates within one plane at a time--no matter how many planes there are in that dimension. Let's go through the process to get the rotational matrices for a 3-dimensional figure.

In the 3rd dimension there are (3(3-1)/2) or 3 planes. This means that, to be able to rotate a figure completely in the 3rd dimension, we will have to set up 3 separate matrices. This will be the X-Y, X-Z, and the Y-Z matrices.

Notice that we have already solved the matrix for the X-Y plane on the previous pages, namely

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

But, in the 3rd dimension we have 3 coordinates--X,Y, and Z. <u>When we rotate within the X-Y plane, we do not alter the value of the Z coordinate</u>. Be sure to visualize this before going further. Similarly when we rotate within the X-Z plane, we do not alter the value of the Y coordinate; and once again, when we rotate within the Y-Z plane, we do not alter the value of the X coordinate.

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

Since we have 3 coordinates in the 3rd dimension, we will need to use a rotational matrix which we can multiply by a matrix containing the values of the X,Y and Z coordinates and come out with a matrix containing our new point X',Y' and Z'. This means that we need a rotational matrix 3 by 3. Another stipulation is that we do not want to alter the 3rd coordinate as we rotate within one plane. For example, when rotating within the X-Y plane we want to leave the Z coordinate alone--unchanged. Referring to our knowledge of matrices we see that for our X-Y plane our matrix should look like this:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As an example:

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} = \begin{bmatrix} x' & y' & z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that the Z value remains unchanged.

Now for the rotational matrix for the X-Z plane. This time we want to leave the value of the Y coordinate unchanged. Again the matrix should be 3 by 3 and should look like the following in order not to change the Y coordinate:

$$\begin{bmatrix} - & 0 & - \\ 0 & 1 & 0 \\ - & 0 & - \end{bmatrix}$$

Without going through the derivation, the complete X-Z matrix should be:

Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Similarly, the Y-Z rotational matrix would be:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

If one wishes to do the derivation of these matrices, all that is necessary is to go through the derivation for the X-Y rotational matrix--but label the plane the X-Z and the Y-Z instead.

Suppose one wanted to rotate a figure A degrees in the X-Y plane, B degrees in the X-Z plane and C degrees in the Y-Z plane; how would one go about this?

One straightforward way would be to take a point (x,y,z) and multiply it by the X-Y rotational matrix using $\theta = A$ to get a new point (x',y',z'). Then take the new point (x',y',z') and multiply it by the X-Z rotational matrix using $\theta = B$ to get the point (x",y",z"). And finally, take the new point (x",y",z") and multiply it by the Y-Z rotational matrix using $\theta = C$ giving us the desired point. Be sure each rotational matrix has the appropriate value for θ; X-Y use $\theta = A$, X-Z use $\theta = B$, and Y-Z use $\theta = C$.

We can also shorten this procedure by finding one matrix--call it Q--which would also be a 3 by 3 matrix which when multiplied by our point (x,y,z), gives the final point (x',y',z'). Obviously, this matrix must contain all the angles A, B, and C. We can get this matrix by multiplying all

Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

three matrices together--two at a time

Here is how it is done. Let

$$\begin{bmatrix} \text{Matrix W} \end{bmatrix} = \begin{bmatrix} \text{Rotational} \\ \text{matrix} \\ \text{X-Z} \end{bmatrix} \begin{bmatrix} \text{Rotational} \\ \text{matrix} \\ \text{Y-Z} \end{bmatrix} .$$

Thus,

$$\begin{bmatrix} \text{Final} \\ \text{Matrix Q} \end{bmatrix} = \begin{bmatrix} \text{Rotational} \\ \text{matrix} \\ \text{X-Y} \end{bmatrix} \begin{bmatrix} \text{Matrix W} \end{bmatrix} .$$

Now, if we had a point (x,y,z), we could get to (x',y',z') by the following method:

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \text{Matrix Q} \end{bmatrix} .$$

One can multiply all 3 rotational matrices together as shown above in the computer program, or a space saving method involves working out by hand the final matrix Q. A listing for filling the final matrix Q with the appropriate operations (sines and cosines, etc.) is given below. X2, Y2, and Z2 are the "θ"s in radians for the Y-Z, X-Z, and X-Y planes respectively.

```
6000  REM    MAT ROTATE
6010  MAT Q=ZER
6020  Q[1,1]=COS(Z2)*COS(Y2)
6030  Q[2,1]=-1*SIN(Z2)*COS(Y2)
6040  Q[3,1]=-1*SIN(Y2)
6050  Q[1,2]=COS(Z2)*(-1)*SIN(X2)*SIN(Y2)+SIN(Z2)*COS(X2)
6060  Q[2,2]=SIN(Z2)*SIN(X2)*SIN(Y2)+COS(Z2)*COS(X2)
6070  Q[3,2]=-1*SIN(X2)*COS(Y2)
6080  Q[1,3]=COS(Z2)*COS(X2)*SIN(Y2)+SIN(Z2)*SIN(X2)
6090  Q[2,3]=-1*SIN(Z2)*COS(X2)*SIN(Y2)+COS(Z2)*SIN(X2)
6100  Q[3,3]=COS(X2)*COS(Y2)
6110  RETURN
```

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

We can now proceed to the rotational matrices needed for rotation in the 4th dimension.

Let us start by listing the various properties which come about in the 4th dimension.

A point needs 4 coordinates (x,y,z,q) to be described precisely.

There are 6 separate planes in our 4-dimensional coordinate system; thus we can rotate within 6 planes.

Immediately we can see that the rotational matrices must be 4 by 4 in order to satisfy the equation we want to set up to get our new point (x', y', z', q'), as shown below:

$$\begin{bmatrix} x' & y' & z' & q' \end{bmatrix} = \begin{bmatrix} x & y & z & q \end{bmatrix} \begin{bmatrix} \text{4 by 4} \\ \text{rotational matrix} \end{bmatrix}$$

We also know that we will have 6 separate matrices--one for each plane: X-Y, X-Z, X-Q, Y-Z, Y-Q, and Z-Q. After we have the rotational matrices for all 6 planes, we could combine them all into one final matrix as we did in the 3rd dimension.

Let us start with the X-Y plane rotational matrix. Again we want to change only the X and Y coordinates while leaving the Z and Q values unchanged. From what we have learned about 2 and 3-dimensional rotations, we can see that the X-Y rotational matrix looks like this:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

Before continuing, make note of the following abbreviations:

Let $c = \cos$ and $s = \sin$

Similarly,

X-Z rotational matrix ---- $\begin{bmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Y-Z rotational matrix ---- $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

X-Q rotational matrix ---- $\begin{bmatrix} c & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & 0 & c \end{bmatrix}$

Y-Q rotational matrix ---- $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & -s & 0 & c \end{bmatrix}$

Z-Q rotational matrix ---- $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$

By now, the pattern becomes evident. Put a "1" in the diagonal of the row and column whose value is not to be changed. Then fill the rest of that row and column with "0"s. Fill the remaining 4 entries with $\cos\theta$, $\sin\theta$, $-\sin\theta$, and $\cos\theta$ respectively.

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

In order to fully rotate a figure in the 4th dimension, we need to specify the degrees of rotation in each of the 6 planes. This means that we must specify 6 angles of rotation. Label these A,B,C,D,E, and F.

This means that we must replace the appropriate value (A,B,C,D,E, or F) in the θ of the corresponding rotational matrix. Thus, as an example, the Y-Q rotational matrix would look like:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos E & 0 & \sin E \\ 0 & 0 & 1 & 0 \\ 0 & -\sin E & 0 & \cos E \end{bmatrix}$$

In order to rotate the point (x,y,z,q) to its new location (x',y',z',q') we need to multiply the original point by each of the rotational matrices. The easiest way is to multiply all the rotational matrices together, in the computer program, to get the final matrix once all the angles A through F are known. We need to use some 'dummy' matrices (matrices J and K) to perform the chain multiplication since most computer systems will multiply only two matrices at a time as shown below:

$$\begin{array}{lccll} \text{Let} & [J] & = & [X\text{-}Y] & [X\text{-}Z] \\ & [K] & = & [Y\text{-}Z] & [J] \\ & [J] & = & [X\text{-}Q] & [K] \\ & [K] & = & [Y\text{-}Q] & [J] \\ & [J] & = & [Z\text{-}Q] & [K] \end{array}$$

Matrix J is now the final rotational matrix. We can get the point (x', y',z',q') in one easy step from the original point (x,y,z,q). This is shown here:

$$\begin{bmatrix} x' & y' & z' & q' \end{bmatrix} = \begin{bmatrix} x & y & z & q \end{bmatrix} \begin{bmatrix} \text{Final} \\ \text{matrix J} \end{bmatrix}$$

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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

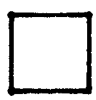
Suppose one wanted to display an n-dimensional figure on a screen--a 2-dimensional object. Our screen can only plot in 2 dimensions--X and Y. So how do we display the n-dimensional figure after we have completed all operations such as scaling, rotations, perspective, etc. on the figure?

Simply plot the X and Y coordinates for each n-dimensional point. On the next page are various graphs of 4-dimensional cubes. The corresponding table shows the amount of rotation within each plane for each graph. In this program all 16 vertices of the 4-dimensional cube are solved for in 4-dimensional parameters--coordinates, matrices, etc. After arriving at an (x',y',z',q') for each vertice, plot the x' and y' values on the screen. The results are shown on the following page.

Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

4-DIMENSIONAL CUBES

Graph #	Degrees turn around the						
	ZQ	YQ	YZ	XQ	XZ	XY	Planes
1	0	0	0	0	0	0	
2	45	0	45	0	45	0	
3	45	45	45	45	45	45	
4	30	40	50	60	70	80	
5	45	0	30	160	225	70	
6	45	30	30	160	225	70	
7	40	80	120	160	200	240	
8	30	75	120	165	205	255	
9	30	30	30	30	30	30	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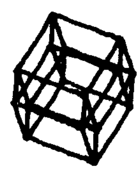
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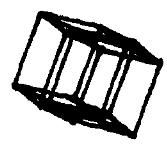
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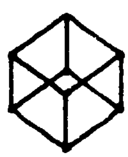
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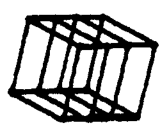
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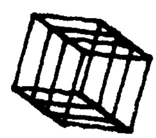
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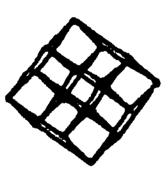
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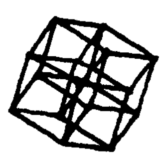
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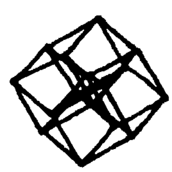
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Mark Gottlieb
4342 Sunset Beach Dr. NW
Olympia, WA 98502

By now the pattern for the rotational matrices for the nth dimension should be obvious.

Briefly, here is the process for the 5-dimensional matrices:

First, we know that we will have 5 coordinates. Let us call them X,Y, Z,Q, and V. To find out how many matrices we must solve for, we first need to know the number of different planes--X-Y, X-Z,...Z-V, and Q-V. The number of planes is (5(5-1)/2) which is 10. This is a large number with which to work, nevertheless, by following the previous procedures, we can easily rotate 5-dimensional figures. The rotational matrices will be of the form shown below.

$$\text{X-Y} \quad \text{-----} \quad \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\text{Q-V} \quad \text{-----} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Rotations in 2 and 3 dimensions will prove very practical while rotations in higher dimensions will prove fascinating as well as mind-boggling. Experiment and be the first one on your block to rotate an imaginary 5-dimensional cube. Or steer a space ship through 4-dimensional space!

###

ADDENDUM

Although it was not mentioned in this text, the following property of matrix multiplication and its consequences should be noted.

Matrix multiplication is not communitive. In other words, $[A][B] \neq [B][A]$ for arbitrary matrices A and B. Consequently, rotations are not communitive. A rotation of 75° within the YZ plane followed by a rotation of 50° within the XZ plane <u>is not equivalent to</u> a rotation of 50° within the XZ plane followed by a rotation of 75° within the YZ plane.

Since the order inwhich rotations are performed affects the final result -- we must be careful to multiply our matrices also in the correct order.

EXAMPLE SHOWING ROTATION OF A BOOK